Stability of particle rotation in a rotating electric field

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Depending on the parameters of a particle and of a host medium, a particle in a rotating electric field can rotate either in the direction of rotation of electric field or in the opposite direction. There exists a range of parameters where both regimes of rotation can be realized simultaneously. In this study we investigate stability of rotation regimes in the whole range of parameters. We determine the range of parameters where different modes of particle rotation are stable, the range of parameters where only one regime of rotation is stable and the range of parameters where both regimes of rotation are unstable. It is shown that if the mode of rotation is realized outside the domain of the existence of two regimes of rotation, then this rotation mode is stable. Consequently, unstable regimes can be realized only in the range of parameters where both modes of rotation can exist simultaneously. We suggest a simple algorithm for determining the range of stability through the parameters of the system which does not require calculating frequencies of particle rotation in each of the possible rotation regimes.

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I. INTRODUCTION

Dynamics of solid or liquid particles in a host medium under the action of an external electric field attracted much attention lately because of numerous technological applications, e.g., manipulation of microparticles in biotechnology and genetic engineering, nanotechnology, and noncontact measurements of physical properties of particles. One of the topics of these theoretical and experimental studies is rotation of liquid or solid particles embedded into a weakly conducting host medium [1–4].

In our previous study [5] we investigated rotation of particle in a rotating electric field. In Ref. [5] it was shown that depending upon permittivity ε_1 and conductivity σ_1 of the host medium and electric conductivity σ_2 and permittivity ε_2 of the particle, under the action of a rotating electric field, the particle immersed in a host medium can rotate either in the direction of rotation of electric field or in the opposite direction. The first mode of rotation occurs when $\sigma_1/\sigma_2<\varepsilon_1/\varepsilon_2$ while the second mode of rotation is realized when

$$\sigma_1/\sigma_2 > \varepsilon_1/\varepsilon_2. \tag{1}$$

In the case when condition (1) is satisfied the particle rotates against the direction of rotation of the external electric field. The mechanism of rotation is associated with a finite time of relaxation of the electric charge $\tau_0=\varepsilon_0\varepsilon_1/\sigma_1$, where ε_0 is permittivity of the vacuum. It is natural to classify particles which satisfy a condition (1) as negative electroviscosity (NEV) particles, and particles which meet the opposite inequality can be classified as positive electroviscosity (PEV) particles. The important difference between the NEV particles and the PEV particles is that in the whole range of parameters the PEV particles rotate in the direction of rotation of the external electric field while for the NEV particles there exists a range of parameters (amplitude and frequency of rotation of the external electric field) whereby NEV particles can rotate either in the direction of rotation of the external electric field or in the opposite direction. To put it differently, the NEV particles can be either in the NEV or PEV regime. The range of parameters where both regimes can be realized simultaneously can be characterized by the frequency of rotation of the electric field $\nu_c(E)$, where E is the amplitude of the external electric field, such that when the frequency $\nu<\nu_c(E)$, both regimes PEV and NEV can be realized simultaneously. In Ref. [5] this range of parameters was called, not quite adequately, the bistable electroviscosity region although the analysis of the range of instability was not performed. In this study we distinguish between the range of parameters where only one regime of rotation can be realized, single electroviscosity (SEV) region, and the range of parameters whereby different regimes of rotation are possible, bielectroviscosity (BEV) region. We investigate the BEV region and distinguish between three types of domains where both regimes are stable, one regime is stable or both regimes are unstable. We show that the domain where both regimes are unstable exists only in the BEV region while rotation regime in the SEV region is always stable.

It must be noted that formally the investigated model describes not only the behavior of the individual particle subjected to the rotating electric field but also the behavior of the sheared fluid-particle suspension in the constant electric field [6]. In Ref. [6] the behavior of particle in the bistability range was analyzed in the case when parameters were selected such that both rotation regimes were stable. It was found that random velocity fluctuations result in an uncompensated flow regime whereby the number of particles rotating in the direction of shear flow vorticity is larger than the number of particles rotating in the opposite direction. Formally, the behavior of particles with angular velocity directed as the vector of the shear flow vorticity vector is equivalent to the behavior of particles rotating against the direction of rotation of the external field. The latter assertion can be easily validated by using the frame of reference that

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rotates with the electric field. Consequently, the behavior of the NEV particles in the NEV regime is equivalent to the behavior of rotating particles having the angular velocity vector directed as the vorticity vector of a shear flow and subjected to the constant electric field. In Refs. [7–9] it was established that in this regime particles reduce the effective viscosity of the suspension while rotating particles with the angular velocity vector directed against the shear flow vorticity vector increase the effective viscosity of the suspension.

Rotation of particles with angular velocity vector directed against the shear flow vorticity vector and subjected to the constant electric field is formally equivalent to the behavior of the NEV particles in the PEV regime under the action of the rotating electric field. Note also that in Ref. [6] the analysis of the bistability domain was conducted for the strength of the electric field $E^2/E_c^2=2$, where E_c is a threshold amplitude of the electric field required for the excitation of Quincke rotation. In the present study we demonstrate that when condition $E^2<8E_c^2/3$ is satisfied, the NEV and the PEV regimes are stable irrespective of the particle inertia or frequency of rotation of the electric field. The obtained results allow expanding the range of parameters and investigating the dependence of the effect of particle velocity fluctuations upon the deviation of the electric field strength from the threshold value. On the other hand, from the theoretical and experimental viewpoint the direct investigation of the behavior of particle suspension under the action of electric field is more involved than investigating a single rotating particle (body). Consequently, the results obtained in the present study are of interest by themselves for validating different physical models in this field. It must be noted that a comprehensive theoretical and experimental investigation of Quincke rotation of a rigid cylinder immersed in a liquid in the case of constant electric field was performed in Ref. [10]. The electric field in this study was directed perpendicular to the rotation axis. The authors investigated stability of rotation of a cylinder under the action of a constant electric field, showed the similarity between this system and Lorenz model [11] and demonstrated a chaotic character of cylinder rotation in the range of the parameters where rotation of the cylinder becomes unstable. In the present study we investigate theoretically the dynamics of the body rotating around its axis of symmetry under the action of electric field which is directed perpendicular to the axis of rotation (see Fig. 1). As an example which illustrates the obtained results we analyze the change of the pattern of rotation of a system with the parameters identical to those considered in Ref. [10].

The main effect of the rotating electric field on the system is associated with elimination of degeneration in the case of a constant electric field where rotations of a cylinder in two opposite directions are physically equivalent. Rotation of the electric field eliminates this degeneration, and behavior of the bodies rotating in opposite directions, for a fixed direction of rotation of the external electric field, is different, e.g., domains of rotation stability are different. Investigation of the domains of stability of different regimes of rotation in the whole range of parameters is the goal of this study.

In Sec. II of this study we describe a mathematical model and in order to make the paper self-contained we present a solution of the problem which has been derived in Ref. [5]. In Sec. III we conduct a complete stability analysis of the solutions obtained in Sec. II and demonstrate the obtained results for the system with the parameters identical to those studied in Ref. [10]. Finally, in Sec. IV we summarize the obtained results.

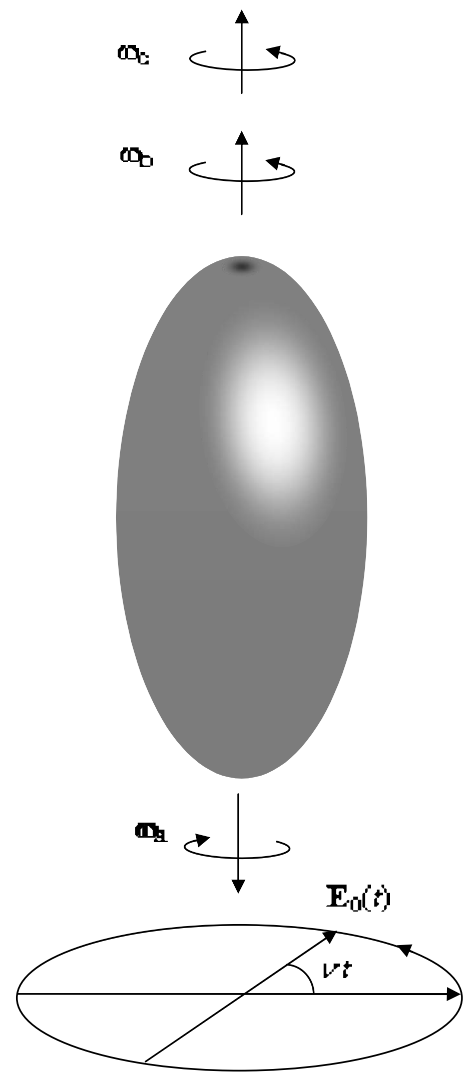

FIG. 1. Spinning of axially symmetric body under the action of electric field $\vec{E}_0$ which rotates in plane perpendicular to axis of symmetry. In the base regime *a* angular velocity is directed opposite direction to the angular velocities in the unstable rotation regimes *b* and additional regime *c*.

II. MATHEMATICAL MODEL

Consider a system consisting of an axially symmetric particle with permittivity ε_2 and electric conductivity σ_2 that is immersed into a liquid or gaseous host medium with permittivity ε_1 and electric conductivity σ_1. The system is subjected to the external electric field with an amplitude E_0 which rotates with a constant angular velocity $\bar{\nu}$:

$$\vec{E}=E_0\sum_{\alpha=\pm 1}\vec{u}_\alpha \exp(-i\alpha\bar{\nu}t), \quad \vec{u}_\alpha=\vec{e}_1+i\alpha\vec{e}_2, \tag{2}$$

where $\vec{e}_1$ and $\vec{e}_2$ are unit Cartesian vectors. The dynamics of particle rotation around the axis of symmetry is determined by the following equation:

$$I\frac{d\vec{\Omega}}{dt}=-\xi\vec{\Omega}+\vec{P}\times\vec{E}, \tag{3}$$

where I is a moment of inertia with respect to the particle's axis, $\vec{\Omega}=\vec{e}_3\Omega(t)$, $\vec{e}_3$ is a unit vector in the direction of the axis of symmetry, Ω is angular velocity, ξ is a rotational friction coefficient which is related with the viscosity of a host medium η, $\xi=f_\xi \eta V$, and f_ξ is a numerical coefficient which depends upon the particle shape and V is a particle volume.

The effective dipole moment of the particle $\vec{P}$ is determined by the following equation [5]:

$$\frac{\partial\vec{P}}{\partial t}-\vec{\Omega}\times\vec{P}+\frac{\vec{P}}{\tau_m}=\frac{\kappa_\sigma-\kappa_\varepsilon}{(1+f_\sigma)(1+f_\varepsilon)}\frac{\vec{P}_s}{\tau_m}, \tag{4}$$

where $\tau_m=\tau_0(1+f_\varepsilon)/(1+f_\sigma)$, $\vec{P}_s=\varepsilon_0\varepsilon_1 V\vec{E}(t)$, $f_\varepsilon=\kappa_\varepsilon(1-n)/2$, $f_\sigma=\kappa_\sigma(1-n)/2$, $\kappa_\varepsilon=\varepsilon_2/\varepsilon_1-1$, $\kappa_\sigma=\sigma_2/\sigma_1-1$, and $\tau_0=\varepsilon_0\varepsilon_1/\sigma_1$.

The system of equations (1)–(3) has a stationary solution which is written below using the dimensionless variables $\vec{X}=\vec{E}/\bar{E}$, $\bar{E}=\sqrt{\xi/(\tau_m V)}$, $\vec{\Pi}=\vec{P}/(\bar{E}V)$, $\omega=\Omega\tau_m$, $\nu=\bar{\nu}\tau_m$:

$$\Pi_\alpha^0=\frac{1-i\alpha(\omega_0-\nu)}{1+(\omega_0-\nu)^2}\frac{\chi X_0}{2}, \tag{5}$$

where frequency $\omega_-=\omega_0-\nu$ is the solution of a cubic equation

$$\omega_-^3+\nu\omega_-^2+(1-\chi X_0^2)\omega_-+\nu=0 \tag{6}$$

and $X_0=E_0/\bar{E}$, $\chi=\varepsilon_0\varepsilon_1(\kappa_\varepsilon-\kappa_\sigma)/[(1+f_\varepsilon)(1+f_\sigma)]$.

The dipole moment of the particle is determined by the following formula:

$$\Pi(\tau)=\sum_{\alpha=\pm 1}\Pi_\alpha^0\exp(-i\alpha\nu\tau), \quad \tau=\frac{t}{\tau_m}. \tag{7}$$

Depending on the magnitude of the parameter χX_0^2 and electric field rotation frequency ν, Eq. (6) has either one or three real roots. Hereafter we investigate the case of NEV particle with the parameters satisfying condition (1), so that $\chi>0$. Equation (6) for rotation frequency $\nu=0$ implies that in addition to the solution $\omega_-=\omega_0=0$ there also exist solutions $\omega_0=\pm\sqrt{\chi X_0^2-1}$. When condition $E_0>E_c=\bar{E}/\sqrt{\chi}$ is met, the particle state of rest, $\omega_0=0$, loses stability, and the particle begins to rotate with a constant angular velocity ω_0 which is indicated above. Consider a case when frequency of rotation of the external electric field $\nu\neq 0$. Let us define the following parameters (see Ref. [5]):

$$R=\chi X_0^2=E_0^2/E_c^2, \quad a_1=9(1+R/2), \quad a_2=3(1-R),$$
$$d_1=\nu(\nu^2+a_1), \quad d_2=|a_2-\nu^2|. \tag{8}$$

When $R<1$, Eq. (6) has only one root. If $\nu^2<a_2$ the root ω_- is determined by the following formula:

$$\omega_-=y_1-\frac{\nu}{3}, \quad y_1=\frac{1}{3}\sum_{\alpha=\pm 1}\alpha(\sqrt{d_1^2+d_2^3}-\alpha d_1)^{1/3}. \tag{9}$$

When the frequency of rotation of the external electric field $\nu=0$, then $d_1=0$ and the body remains in the state of rest $(y_1=0)$. When the frequency of rotation differs from zero, the body rotates in the main regime against the direction of rotation of the external electric field ($\chi>0$, see Ref. [5]).

The second domain is determined by the relations $\nu^2>a_2$ and $|d_1|>\sqrt{d_2^3}$. There is also one root in this domain:

$$\omega_-=y_2-\frac{\nu}{3}, \quad y_2=-\frac{1}{3}\,\mathrm{sgn}(d_1)\sum_{\alpha=\pm 1}(|d_1|+\alpha\sqrt{d_1^2-d_2^3})^{1/3}. \tag{10}$$

This domain also corresponds to a SEV region when the body rotates in the main regime against the direction of rotation of the external electric field.

The third domain with three real roots is determined by the relations $\nu^2>a_2$ and $|d_1|<\sqrt{d_2^3}$. Formulas for the roots in this domain $y_a<y_b<y_c$ read

$$y_a=-2\cos(\varphi_0), \quad y_b=2\cos(\varphi_0+\pi/3),$$
$$y_c=2\cos(\varphi_0-\pi/3), \quad \varphi_0=\frac{1}{3}\cos^{-1}\left(\frac{d_1}{\sqrt{d_2^3}}\right). \tag{11}$$

Formula for the relative particle rotation frequency in this domain ω_- reads

$$\omega_-^k=-\frac{\nu}{3}+\frac{\sqrt{d_2}}{3}y_k, \tag{12}$$

where $k=a,b,c$.

This domain corresponds to a BEV region when apart from the main regime of rotation there appear additional regimes whereby a particle rotates in the direction of rotation of the external electric field. When the frequency of rotation of the external electric field $\nu>0$, the rotation regime a corresponds to the main regime while additional regimes b and c appear only in the amplitude range $R>1$. Furthermore, we will show that regime b is always unstable. When direction of rotation of the external field is changed to the opposite (that corresponds to the substitution $\nu\to-\nu$) the main regime of rotation is determined by c branch of the solution while a branch describes the additional solution which appears in the amplitude range $R>1$. Denote by ϕ^k some characteristics of the system in the regime k, e.g., particle rotation frequency, increment of instability growth, range of stable rotation, etc. Then ϕ^a is replaced by ϕ^c, when ν is changed to $-\nu$ and vice versa.

In Fig. 2 we showed the dependence of $\omega_-^k(\nu,R)$ vs rotation frequency of the external electric field ν for $R=3$, $R=5$ and various solutions $k=a,b,c$. Equations (8)–(12) were

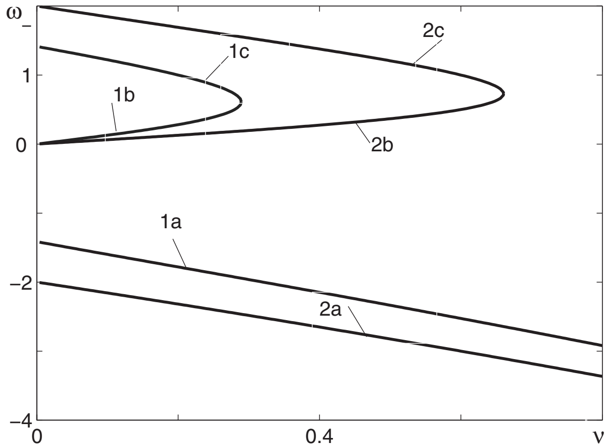

FIG. 2. Dependence of the rotation frequency ω_- vs rotation frequency of the external electric field ν in the BEV domain for three rotation regimes ω_-^a, ω_-^b, ω_-^c. (1a), (1b), (1c): $R=3$; (2a), (2b), (2c): $R=5$.

derived in Ref. [5]. In the latter study we also determined the critical magnitude of the rotation frequency $\nu_c(R)$ which separates between the domains with various regimes of rotation and domains where only one basic regime of rotation can be realized. Formula for the critical rotation frequency $\nu_c(R)$ reads

$$\nu_c = \frac{\sqrt{R^2 - 20R - 8 + \sqrt{R(8+R)^3}}}{2\sqrt{2}}. \tag{13}$$

Formula (13) is equivalent to formula (19) in Ref. [5]. The inverse of the function $\nu_c(R)$, $R=R_c(\nu)$, satisfies a cubic equation and determines the amplitude of the external field for given ν. Equation for $R_c(\nu)$ reads

$$(R_c-1)^3 + \frac{\nu^2}{4}(R_c-1)^2 - \frac{9\nu^2}{2}(R_c-1) - \frac{\nu^2}{4}(27+4\nu^2) = 0. \tag{14}$$

For a given frequency ν the BEV domain is realized when the frequency of rotation of the external electric field $\nu < \nu_c(R)$ or if $R > R_c(\nu)$.

We do not present here an explicit formula for $R_c(\nu)$. Note only that for small rotation frequencies of the external electric field $\nu \ll 1$, $R_c = 1 + 3|\nu|^{2/3}/4$, and $R_c \approx \nu$ for $\nu \to \infty$.

Consequently, to the bifurcation point $R=R_c=1$ which separates between the domain where a body is at rest and a rotation domain for $\nu=0$, corresponds a curve $R=R_c(\nu)$ when $\nu \neq 0$ (see Fig. 6, curve 1). This curve separates between the domain where a body rotates against the direction of rotation of the electric field (SEV region) and the domain where depending on initial conditions a body can rotate in different directions (BEV region). In the next section we investigate BEV and SEV domains with respect to the stability of the regimes which are realized in these domains.

III. STABILITY OF PARTICLE ROTATION REGIMES

Using a standard procedure for investigating dynamic stability, we seek for the solution of Eqs. (3) and (4) in the vicinity of a solution (5)–(7) in the following form:

$$\Pi_\alpha^k = \Pi_{0\alpha}^k + \exp(\gamma t)\Pi_{1\alpha}^k, \quad \omega^k(t) = \omega_0^k + \exp(\gamma t)\omega_1^k,$$

$$\Pi_\alpha = (\vec{\Pi} \cdot \vec{u}_\alpha^*)\exp(i\alpha\nu t),$$

where k denotes different solutions of Eqs. (5) and (6). Using this procedure and the Routh-Hurwitz stability condition $\mathrm{Re}(\gamma) < 0$ (see, e.g., Ref. [12]), yields the following inequalities:

$$\Delta_1(Z_k) = Z_k^2 + 2p(1 + p/2 - R/4)Z_k - Rp^2/2 > 0, \tag{15}$$

$$\Delta_2(Z_k) = Z_k^2 + RZ_k - 2R > 0, \tag{16}$$

where

$$Z_k = 1 + (\omega_-^k)^2, \quad p = \xi\tau_m/I. \tag{17}$$

Conditions (15) and (16) are the necessary and the sufficient conditions for $\mathrm{Re}(\gamma) < 0$ when the magnitude of $Z_k(\nu,R)$ is given. Alternatively, considering $\Delta_1(Z)$ and $\Delta_2(Z)$ as quadratic polynomials with respect to Z and taking into account that we are interested in the range $Z > 1$, we obtain the conditions for the positivity of $\Delta_1(Z)$ and $\Delta_2(Z)$. Condition (15) is satisfied when $Z_k(\nu,R) > Z_1(p,R)$, where

$$Z_1(p,R) = p[\sqrt{(1+p/2-R/4)^2 + R/2} - (1+p/2-R/4)]. \tag{18}$$

Condition (16) is satisfied when $Z_k(\nu,R) > Z_m(R)$, where

$$Z_m(R) = R(\sqrt{1+8/R} - 1)/2. \tag{19}$$

Equations (6) and (17) yield the following relation:

$$\nu^2 = (Z-1)(R-Z)^2/Z^2. \tag{20}$$

Equation (20) is a cubic equation with respect to the parameter Z and determines three functions $Z_k(\nu,R)$ which correspond to three branches of solutions of this equation. In Fig. 3 these branches are denoted by numbers 1, 2, 3. For $\nu > 0$ the solutions a,c,b correspond to these branches. Function $\nu = \pm\sqrt{\nu^2(R,Z)}$ which is determined by Eq. (20) is an inverse function of $Z_k(\nu,R)$. Function $Z_k(\nu,R)$ is shown in Fig. 3 for

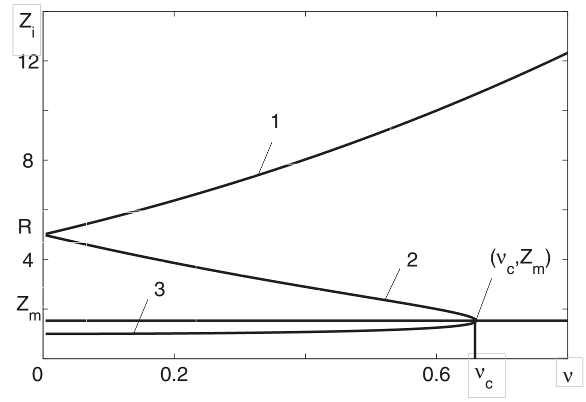

FIG. 3. Dependence of parameter Z for different branches of particle rotation regimes vs amplitude of the rotating electric field. $1: Z=Z_a(\nu,R)$, $2: Z=Z_c(\nu,R)$, $3: Z=Z_b(\nu,R)$, $R=5$, and $\nu > 0$.

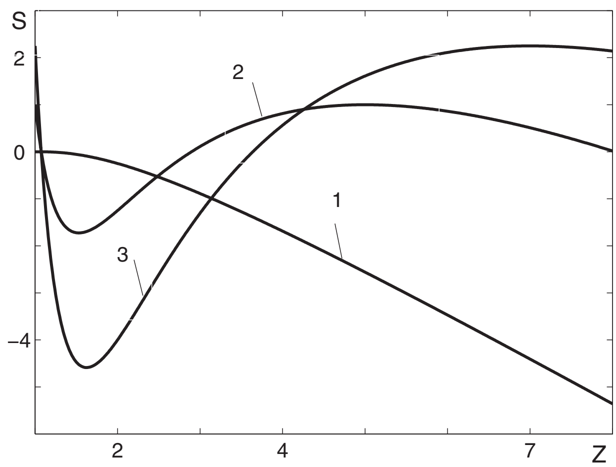

FIG. 4. Dependence $S=S(\nu,R,Z)$ vs Z for different values of parameters R,ν. $1:\nu=0$, $R=1$, $2:\nu=1$, $R=5$, $3:\nu=1.5$, $R=7$.

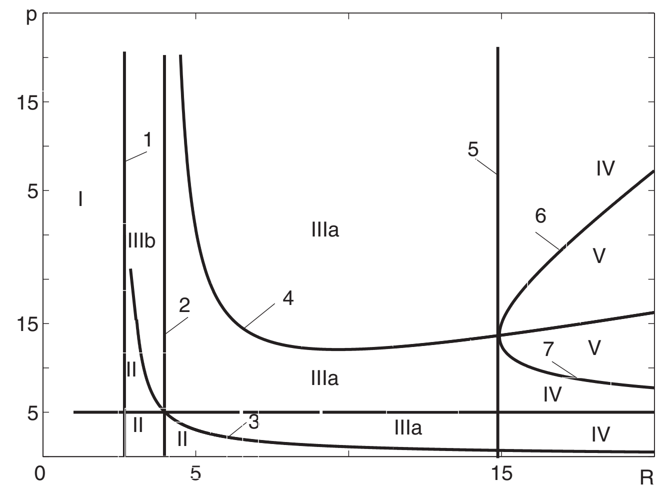

FIG. 5. Subdomains of the different stability in the (p,R) plane. Curve $1:R=8/3$, curve $2:R=4$, curve $3:p=p_1(R)$, curve $4:p=p_{\max}(R)$, curve $5:R=8+2\sqrt{12}$, curve $6:p=p_+(R)$, curve $7:p=p_-(R)$. Subdomain I and subdomain II: The rotation regimes a and c are stable irrespective of the frequency of rotation of the external field. Subdomain IIIa: Regime a is stable and regime c is stable for frequency $\nu<\nu^c_{\mathrm{cr,min}}(R)$ irrespectively of the magnitude of the parameter p. Subdomain IIIb: $1<R<4$, the minimum frequency $\nu^c_{\mathrm{cr,min}}(R)$ does not exist and stability of regime c depends upon parameter p. Subdomain IV: Regime a is stable and stability of the regime c depends upon parameters p,ν. Subdomain V: The regime c is unstable and the regime a is stable when $\nu>\nu^a_{\mathrm{cr,max}}(R)$.

$R=5$ and $\nu>0$. Analysis of cubic equation (20) yields the following inequalities for functions $Z_k(\nu,R)$ for $\nu>0$:

$$Z_a(\nu,R) > R > Z_c(\nu,R) > Z_m(R) > Z_b(\nu,R). \qquad (21)$$

The point $Z=R$, $\nu=0$ is an intersection point of the branches a and c. Consequently, the domain of stability of branches a and c coincide only in the case of a constant electric field while rotation of the external electric field violates the symmetry. The latter causes, as we will show further, expansion of the domain of stability of the regime whereby a particle rotates against the direction of rotation of the external field, and contraction of the domain of stability of the regime whereby a particle rotates in the direction of rotation of the external field. The point $Z=Z_m(R)$, $\nu=\nu_c(R)$, where $Z_m(R)$ is determined by Eq. (19), is an intersection point for the branches c and b. Consequently, the condition (16) is always satisfied for the branches a and c but is not met for the branch b. Therefore the branch b is always unstable while the condition for stability of branches a and c reads

$$Z_k(\nu,R) > Z_1(p,R), \quad k=a,c. \qquad (22)$$

For further analysis let us introduce the function

$$S(\nu,R,Z) = \nu^2 - (Z-1)(R-Z)^2/Z^2. \qquad (23)$$

This function has a minimum at $Z=Z_m(R)$, a maximum at $Z=R$ and is shown in Fig. 4. In the interval $Z_m(R)<Z<R$ function $S(\nu,R,Z)$ increases monotonically, and $S=0$ at $Z=Z_k(\nu,R)$. Consequently, if $Z_1(p,R)<R$ then a condition of stability $Z_1(p,R)<Z_k(\nu,R)$ implies that $S<0$ for $Z=Z_1(p,R)$. Since in the interval $Z>R$ function $S(\nu,R,Z)$ decreases monotonically, a stability condition in this interval implies that $S>0$ for $Z=Z_1(p,R)$. Therefore assessing the stability of the realized rotation regimes can be performed using Eqs. (18) and (23). The resulting procedure is as follows. The magnitude of function $Z_1(p,R)$ is determined from Eq. (18) for given parameters of the problem p,R,ν. If $Z_1(p,R)<R$ and $\nu>0$, the branch a is always stable and the branch c is stable if and only if $S[\nu,R,Z_1(p,R)]<0$. When $Z_1(p,R)>R$ the branch c is always unstable and the branch a is stable if and only if $S[\nu,R,Z_1(p,R)]>0$.

Hereafter we consider only the case $\nu>0$ since the obtained results can be extended for $\nu<0$ after replacement $a\to c$, $c\to a$ as was mentioned above. Inequality (22) determines an intricately shaped domain in the space of the parameters (ν,p,R). In the following we consider the projections of this domain on the planes (p,R) and (ν,R), see Figs. 5 and 6, respectively. For analysis of this domain let us introduce the following function:

$$\nu^2_{\mathrm{cr}}(p,R) = [Z_1(p,R)-1][R-Z_1(p,R)]^2/Z_1^2(p,R). \qquad (24)$$

Taking into account the behavior of function $S(\nu,R,Z)$ in the interval $Z>R$ and a condition of stability in this range of Z it can be easily seen that Eq. (24) determines such $\nu^a_{\mathrm{cr}}(p,R)=\nu[Z_1(p,R),R]$ that for $\nu>\nu^a_{\mathrm{cr}}$ the branch a is always stable for given p,R. Since $Z_a(\nu,R)>R$ for arbitrary values of ν,R, the conditions of stability of the branch a read

$$\nu > \nu^a_{\mathrm{cr}} = \sqrt{Z_1(p,R)-1}[Z_1(p,R)-R]/Z_1(p,R),$$
$$Z_1(p,R) > R, \qquad (25)$$

and also

$$Z_1(p,R) < R. \qquad (26)$$

The condition of stability (25) determines a lower bound for the rotation frequency of the external electric field. Therefore by increasing the rotation frequency of the external field ν it is always possible to stabilize rotation in regime a. Condition (26) determines the domain where stability of rotation in regime a does not depend upon the frequency of rotation of the external field, ν. On the (p,R) plane these are domains

I–IV in Fig. 5. Only in the domain V, where $Z_1(p,R)>R$, is the condition (26) violated. The curve $Z_1(p,R)=R$ separating between domain V and other domains, consists of two branches $p=p_+(R)$ and $p=p_-(R)$,

$$p_\pm(R) = 2[R/4 - 1 \pm \sqrt{(R/4-1)^2 - R/2}],$$

which correspond to two branches of the equation $R=R_{\rm cr}(p)=p(p+4)/(p-2)$. For $\nu=0$ in the range $R>R_{\rm cr}(p)$ rotation loses stability and becomes chaotic. In Ref. [10] it was shown that when $\nu=0$, the system is similar to that investigated by Lorenz [11] and its behavior has been investigated comprehensively in the literature [13]. Rotation of the external electric field ($\nu\neq 0$) eliminates degeneration, and domains of stability for branches a and c in this case do not coincide. Inequality (21) and above considerations show that in the domain V (see Fig. 5) only branch a of the solution can be stable while branch c is always unstable because $Z_c(\nu,R)<R<Z_1(p,R)$. Taking into account inequality $Z_c(\nu,R)<R$ and behavior of function $S(\nu,R,Z)$ in this range of parameter Z we arrive at the conditions of stability of the branch c:

$$\nu < \nu_{\rm cr}^c = \sqrt{[Z_1(p,R)-1][R-Z_1(p,R)]/Z_1(p,R)},$$
$$Z_1(p,R) < R, \tag{27}$$

and also

$$Z_1(p,R) < Z_m(R). \tag{28}$$

Condition of stability (27) determines an upper bound for the rotation frequency of the external electric field. Consequently, stabilization of the rotation regime c requires the reduction of the rotation frequency of the external electric field. The condition of stability (28) arises because function $Z_c(\nu,R)$ varies in the range $Z_m(R)<Z_c(\nu,R)<R$ and monotonically decreases with ν. Since $Z_c(\nu,R)>Z_m(R)$, condition (28) determines the domain of stability of the branch c independent of the value of parameter ν. In the Appendix we show that this condition is satisfied for $R<8/3$ and also for $R>8/3$, $p<p_1(R)$, where

$$p_1 = d_1/2 + \sqrt{d_1^2/4 + d_2},$$

$$d_1 = 4Z_m(R)(1-R/4)/(R-2Z_m),$$

$$d_2 = 2Z_m^2/(R-2Z_m). \tag{29}$$

These domains are denoted in Fig. 5 as domains I and II. Consequently, in domain I which is separated from other domains by a curve $R=8/3$, rotation regime c is always stable. In domain II, $R>8/3$, $p<p_1(R)$, the rotation regime c is also always stable. In domains IIIa, IIIb, and IV (see Fig. 5) stability of the rotation regime c depends upon the frequency of rotation of the external electric field ν. In domain IIIa, which is realized for the external field strength in the range $4<R<8+2\sqrt{12}$, there exists a minimum value of frequency $\nu_{\rm cr,min}^c$, such that for rotation frequency of the external field $\nu<\nu_{\rm cr,min}^c$ the rotation regime c is always stable independently of parameter p. The latter claim is the consequence of the behavior of $Z_1(p,R)$ as a function of a parameter p for given R. Function $Z_1(p,R)$ attains a maximum $Z_{\max}(R)$, for $p=p_{\max}(R)$, where

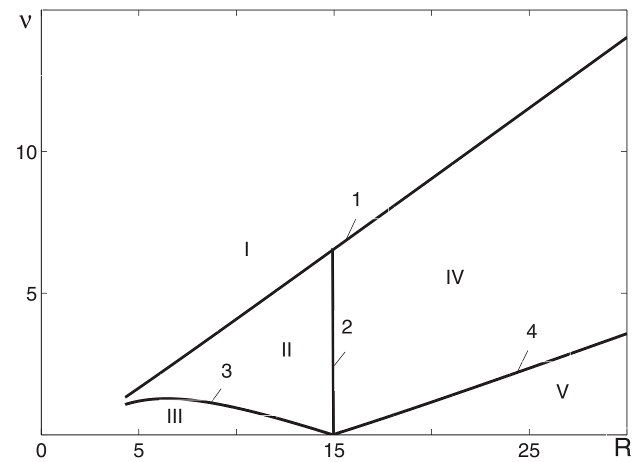

FIG. 6. Subdomains with different stability in the (ν,R) plane. Curve $1: \nu=\nu_c(R)$, curve $2: R=8+2\sqrt{12}$, curve $3: \nu=\nu_{\rm cr,min}^c(R)$, curve $4: \nu=\nu_{\rm cr,max}^a(R)$. Subdomain I: SEV region. Subdomain II: The regime a is stable and stability of the regime c depends upon the parameter p. Subdomain III: The regimes a and c are stable. Subdomain IV: The regime a is stable and stability of the regime c depends upon parameter p. Subdomain V: The regime c is unstable and stability of regime a depends upon the parameter p.

$$p_{\max}(R) = (R^2/4+1)/(R/4-1). \tag{30}$$

In the domain $Z_{\max}(R)=Z_1[p_{\max}(R),R]<R$ or $R<8+2\sqrt{12}$, the maximum value of Z_1 corresponds to the minimum magnitude of $\nu=\nu_{\rm cr,min}^c(R)$. We do not present here the apparent but cumbersome expressions for $Z_{\max}(R)$ and $\nu_{\rm cr,min}^c(R)$ which can be obtained by direct substitution of Eq. (30) in Eqs. (18) and (27), correspondingly. Let us note only that $Z_{\max}(R)\to 2$ for $R\to 4$ and in this case $\nu_{\rm cr}^c\to 1$ (see Fig. 6). In the range $R>8+2\sqrt{12}$, $Z_{\max}(R)>R$. Substituting $Z_{\max}(R)$ in Eq. (25) yields the maximum value of $\nu_{\rm cr}^a$, $\nu_{\rm cr,max}^a(R)=\nu_{\rm cr}[p_{\max}(R),R]$ such that for $\nu>\nu_{\rm cr,max}^a$ the branch a is stable in the whole range of p. In Fig. 6 functions $\nu_{\rm cr,min}^c(R)$ and $\nu_{\rm cr,max}^a(R)$ are plotted as curves 3 and 4, respectively. Finally, in the subdomain V (see Fig. 5), $R>8+2\sqrt{12}$, $p<p_+(R)$, $p>p_-(R)$, $Z_1(p,R)>R>Z_c(\nu,R)$, branch c is always unstable. Branch a is stable everywhere except for the subdomain V where it is stable irrespectively of the parameter p when $\nu>\nu_{\rm cr,max}^a$.

In Fig. 6 we showed different stability domains on the plane (ν,R). The curve 1 which is determined by Eq. (13) separates between the subdomain I where only the main regime (SEV region) is realized and subdomains where different rotation regimes can be realized simultaneously. The curve 2 corresponds to $R=8+2\sqrt{12}$, curves 3, 4 correspond to the functions $\nu=\nu_{\rm cr,min}^c$ and $\nu=\nu_{\rm cr,max}^c$. The main regime a is stable everywhere irrespective of the parameter p except for the domain V where stability of the regime a depends upon the particle's inertia which is determined by the parameter p. Regime c can be stable only in the subdomains II, III, IV. In the subdomain III the regime c is stable independent of the parameter p. In the subdomain V the regime c is always

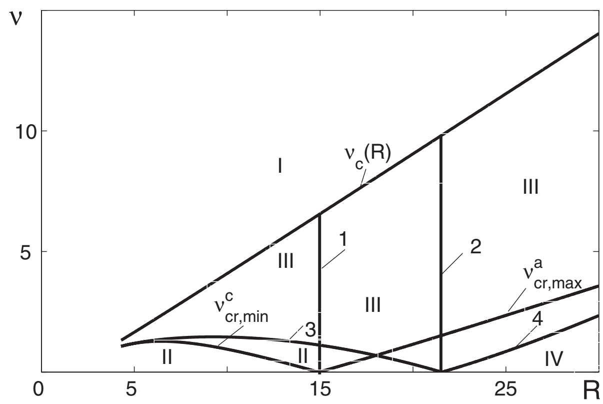

FIG. 7. Subdomains with different stability in the (ν,R) plane for $\bar{p}=14.6$. Curve $1: R=8+2\sqrt{12}$, curve $2: R=\bar{R}=21.5$, curve $3: \nu=\nu_{\mathrm{cr}}(\bar{p},R)\geq\nu^{c}_{\mathrm{cr,min}}$, $4<R<8+2\sqrt{12}$, curve $4: \nu=\nu_{\mathrm{cr}}(\bar{p},R)<\nu^{a}_{\mathrm{cr,max}}$, $R>8+2\sqrt{12}$. Subdomain I: SEV region. Subdomain II: The regimes a and c are stable. Subdomain III: The regime a is stable and the regime c is unstable. Subdomain IV: The regimes a and c are unstable.

unstable. Subdomain III resides in the range $4<R<8+2\sqrt{12}$ because for $R<4$ function $Z_1(p,R)$ does not have maximum for given R.

In the above considerations we assumed that $R>1$ and $\nu<\nu_c(R)$. For $R<1$ and $\nu>\nu_c(R)$ only the main regime is realized, and the situation is substantially simplified. Analysis of the function $Z_1(p,R)$ shows that for $R<1$, $Z_1(p,R)<1<Z_k(\nu,R)$, and, consequently, the main regime which is realized in this domain is always stable. Stability of the main regime for $\nu>\nu_c(R)$ is the direct consequence of the inequality $\nu^{a}_{\mathrm{cr,max}}<\nu_c(R)$, and the branch a remains continuous when intersecting with the curve $\nu=\nu_c(R)$.

For illustrating the physical meaning of the obtained results let us consider the dynamics of particle with $p=\bar{p}=14.6$. The latter choice of the parameter corresponds to the parameter p of the dielectric cylinder whose rotation under the action of a constant electric field directed normally to the axis of the cylinder, was investigated in Ref. [10]. Consider the behavior of the cylinder with the increase of R. Hereafter the regime of rotation whereby the body rotates against the direction of rotation of the external fields is called a base regime while the rotation regime whereby directions of rotation of the particle and the field coincide, is called the additional regime (AR). The BR remains stable in the whole range of the electric field strength $E<\sqrt{8+2\sqrt{12}}E_c$. For $E>\sqrt{8+2\sqrt{12}}E_c$, BR remains stable up to the strength of the electric field $\bar{R}=\bar{p}(\bar{p}+4)/(\bar{p}-2)=21.5$. For $R>\bar{R}$ the BR remains unstable until the rotation frequency of the external electric field ν remains less than the critical frequency $\nu^{a}_{\mathrm{cr}}(\bar{p},R)$ which is determined by Eq. (25) with $p=14.6$ (see curve 4 in Fig. 7). In the domain IV stable rotation modes do not exist, and the behavior of the system can be chaotic. If $\nu>\nu^{a}_{\mathrm{cr}}(\bar{p},R)$, the stationary regime is restored, and the particle rotates with the frequency corresponding to the BR. It must be emphasized again that $\nu^{a}_{\mathrm{cr}}(p,R)$ is always smaller than $\nu^{a}_{\mathrm{cr,max}}=\nu^{a}_{\mathrm{cr}}[p_{\max}(R),R]$. This situation is shown in Fig. 7. The particle behavior in the AR can be analyzed similarly. In the interval of the electric field strength $8/3>R>1$ this regime is always stable. When $R>8/3$ and $p<p_1(R)$ the AR remains stable irrespectively of the rotation frequency ν until $R<R'$ such that $\bar{p}=p_1(R')$. For $\bar{p}=14.6$, $R'=2.83$. Beginning from the latter value of R the AR regime is stable if $R<\bar{R}=21.5$ and $\nu<\nu^{c}_{\mathrm{cr}}(\bar{p},R)$, where $\nu^{c}_{\mathrm{cr}}(\bar{p},R)$ is determined by Eq. (27) (see domain II in Fig. 7). It was noted above that function $\nu_{\mathrm{cr}}(p,R)$ has a minimum $\nu=\nu^{c}_{\mathrm{cr,min}}$ only in the interval $4<R<8+2\sqrt{12}$, while outside this interval the minimum does not exist (see Fig. 7).

IV. CONCLUSIONS

We determined subdomains with different dependencies of the criteria of the stability of particle rotation regimes vs parameters of the problem. It was demonstrated that in the subdomain, where several regimes can be realized, these regimes can be either simultaneously stable, simultaneously unstable or only the main rotation regime can be realized. In the SEV domain where only the base regime (BR) is realized, this regime is always stable. A simple algorithm for evaluating stability of the rotation regime based on Eqs. (18) and (23) allows the determination of the stability of the regimes without calculating frequency of particle rotation.

APPENDIX: RANGES OF VALIDITY OF THE CONDITION (26)

In the following we outline the proof that the condition (26)

$$Z_1(p,R) < Z_m(R) \tag{A1}$$

is satisfied only when $R<8/3$ or $R>8/3$, $p<p_1(R)$, where function $p_1(R)$ is determined by Eq. (29).

Condition (A1) implies the following inequalities

$$\Delta_3 = p^2[2Z_m(R) - R] + 4Z_m(R)(1 - R/4)p + 2Z_m^2(R) > 0, \tag{A2}$$

$$\Delta_4 = p^2 + 2(1 - R/4)p + 2Z_m(R) > 0. \tag{A3}$$

If $R<8/3$, then $Z_m>R/2$ and conditions (A2) and (A3) are satisfied irrespectively of $p>0$. When $R>8/3$, the condition (A2) yields

$$p < p_1 = d_1/2 + \sqrt{d_1^2/4 + d_2}, \tag{A4}$$

where $d_1=4Z_m(R)(1-R/4)/(R-2Z_m)$, $d_2=2Z_m^2/(R-2Z_m)$.

Condition (A3) is satisfied always if $R<R_0=4(2\sqrt{2}+\sqrt{3}-1)$. The latter condition is equivalent to $Z_m>(R/4-1)^2/2$. For $R>R_0$, condition (A3) is satisfied if $p>p_2(R)$ or $p<p_3(R)$, where

$$p_{2,3} = R/4 - 1 \pm \sqrt{(R/4 - 1)^2 - 2Z_m}. \tag{A5}$$

In the whole range of parameter R, $p_1(R) < p_3(R)$. Taking into account that Eqs. (A2) and (A3) must be satisfied simultaneously we arrive at the condition of stability of the regime c in the whole domain BV:

$$p < p_1(R). \tag{A6}$$

The subdomain determined by condition (A6) is shown in Fig. 4 (subdomain II). When $p > p_1(R)$, the stability of the branch c depends upon the frequency ν.

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